OMEGA-LIMIT SETS CLOSE TO SINGULAR-HYPERBOLIC ATTRACTORS

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Abstract. We study the omega-limit sets $\omega_X(x)$ in an isolating block U of a singular-hyperbolic attractor for three-dimensional vector fields X. We prove that for every vector field Y close to X the set $\{x \in U : \omega_Y(x)$ contains a singularity$\}$ is *residual* in U. This is used to prove the persistence of singular-hyperbolic attractors with only one singularity as chain-transitive Lyapunov stable sets. These results generalize well known properties of the geometric Lorenz attractor [GW] and the example in [MPu].

1. Introduction

The *omega-limit set* of x with respect to a vector field X with generating flow X_t is the accumulation point set $\omega_X(x)$ of the positive orbit of x, namely

$$\omega_X(x) = \left\{ y : \ y = \lim_{t_n \to \infty} X_{t_n}(x) \text{ for some sequence } t_n \to \infty \right\}.$$

The structure of the omega limit sets is well understood for vector fields on compact surfaces. In fact, the *Poincaré-Bendixon Theorem* asserts that the omega-limit set for vector fields with finite many singularities in S^2 is either a periodic orbit or a singularity or a graph (a finite union of singularities an separatrices forming a closed curve). The *Schwartz Theorem* implies that the omega-limit set of a C^∞ vector field on a compact surface either contains a singularity or an open set or is a periodic orbit. Another result is the *Peixoto Theorem* asserting that an open dense subset of vector fields on any closed orientable surface are *Morse-Smale*, namely their nonwandering set is formed by a finite union of closed orbits all of whose invariant manifolds are in general position. A direct consequence this result is that, for an open-dense subset of vector fields on closed orientable surfaces, most omega-limit sets are contained in the attracting closed orbits. This provides a complete description of the omega limit sets on closed orientable surfaces.

The above results are known to be false in dimension > 2. Hence extra hypotheses to understand the omega-limit sets are needed in general. An important one is the hyperbolicity introduced by Smale in the sixties. Recall that a compact

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invariant set is *hyperbolic* if it exhibits contracting and expanding direction which together with the flow's direction form a continuous tangent bundle decomposition. This definition leads the concept of *Axiom A vector field*, namely the ones whose non-wandering set is both hyperbolic and the closure of its closed orbits. The Spectral Decomposition Theorem describes the non-wandering set for Axiom A vector fields, namely it decomposes into a finite disjoint union of hyperbolic basic sets. A direct consequence of the Spectral Theorem is that for every Axiom A vector field X there is an open-dense subset of points whose omega-limit set are contained in the hyperbolic attractors of X. By *attractor* we mean a compact invariant set Λ which is *transitive* (i.e. $\Lambda = \omega_X(x)$ for some $x \in \Lambda$) and satisfies $\Lambda = \cap_{t\geq 0} X_t(U)$ for some compact neighborhood U of it called *isolating block*. On the other hand, the structure of the omega-limit sets in an isolating block U of a hyperbolic attractor is well known: For every vector field Y close to X the set

$$\{x \in U : \omega_Y(x) = \cap_{t\geq 0} Y_t(U)\}$$

is *residual* in U. In other words, the omega-limit sets in a residual subset of U are uniformly distributed in the maximal invariant set of Y in U. This result is a direct consequence of the structural stability of the hyperbolic attractors.

There are many examples of non-hyperbolic vector fields X with a large set of trajectories going to the attractors of X. Actually, a conjecture by Palis [P] claims that this is true for a dense set of vector fields on any compact manifold (although he used a different definition of attractor). A strong evidence is the fact that there is a residual subset of C^1 vector fields X on any compact manifold exhibiting a residual subset of points whose omega-limit sets are contained in the chain-transitive Lyapunov stable sets of X ([MPa2]). We recall that a compact invariant set Λ is *chain-transitive* if any pair of points on it can be joined by a pseudo-orbit with arbitrarily small jump. In addition, Λ is *Lyapunov stable* if the positive orbit of a point close to Λ remains close to Λ. The result [MPa2] is weaker than the Palis conjecture since every attractor is a chain-transitive Lyapunov stable set but not vice versa.

In this paper we study the omega-limit sets in an isolating block of an attractor for vector fields on compact three manifolds. Instead of hyperbolicity we shall assume that the attractor is *singular-hyperbolic*, namely it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction [MPP1]. These attractors were considered in [MPP1] for a characterization of C^1 robust transitive sets with singularities for vector fields on compact three manifolds (see also [MPP3]). The singular-hyperbolic attractors are not hyperbolic although they have some properties resembling the hyperbolic ones. In particular, they do not have the pseudo-orbit tracing property and are neither expansive nor structural stable.

The motivation for our investigation is the fact that if U is an isolating block of the geometric Lorenz attractor with vector field X then for every Y close to X the set $\{x \in U : \omega_Y(x) = \cap_{t\geq 0} Y_t(U)\}$ is residual in U (this is precisely the

same property of the hyperbolic attractors reported before). It is then natural to believe that such a conclusion holds if U is an isolating block of a singular-hyperbolic attractor. The answer however is negative as the example [MPu, Appendix] shows. Despite we shall prove that if U is the isolating block of a singular-hyperbolic attractor of X, then the following alternative property holds: For every vector field Y C^r close to X the set

$$\{x \in U : \omega_Y(x) \text{ contains a singularity}\}$$

is *residual* in U. In other words, the positive orbits in a residual subset of U look to be "attracted" to the singularities of Y in U. This fact can be observed with the computer in the classical polynomial Lorenz equation [L]. It contrasts with the fact that the union of the stable manifolds of the singularities of Y in U *is not residual in any open set.* We use this property to prove the persistence singular-hyperbolic attractors with only one singularity as chain-transitive Lyapunov stable sets.

Now we state our result in a precise way. Hereafter M denotes a compact Riemannian three manifold unless otherwise stated. If $U \subset M$ we say that $R \subset U$ *residual* if it realizes as a countable intersection of open-dense subsets of U. It is well known that every residual subset of U is dense in U. Let X be a C^r vector field in M and let X_t be the flow generated by X, $t \in I\!\!R$. A compact invariant set is *singular* if it contains a singularity.

Definition 1.1 (Attractor)**.** *An* attracting set *of X is a compact, invariant, non-empty, set of X equals to $\cap_{t>0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is called* isolating block*. An* attractor *is a transitive attracting set.*

Remark 1.2. [Hu] *calls attractor what we call attracting set. Several definitions of attractor are considered in* [Mi]*.*

Denote by $m(L)$ and $Det(L)$ the minimum norm and the Jacobian of a linear operator L respectively.

Definition 1.3. *A compact invariant set Λ of X is partially hyperbolic if there is a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s \oplus E^c$ and positive constants K, λ such that*
1. E^s is contracting: $\|\, DX_t(x)/E^s_x \,\| \leq Ke^{-\lambda t}$, for every $\forall t > 0$ and $x \in \Lambda$;
2. E^s dominates E^c: $\frac{\|DX_t(x)/E^s_x\|}{m(DX_t(x)/E^c_x)} \leq Ke^{-\lambda t}$, for every $\forall t > 0$ and $\forall x \in \Lambda$.
We say that Λ has volume expanding central direction *if*

$$|\, Det(DX_t(x)/E^c_x) \,| \geq K^{-1}e^{\lambda t},$$

for every $t > 0$ and $x \in \Lambda$.

A singularity σ of X is *hyperbolic* if its eigenvalues are not purely imaginary complex number.

Definition 1.4 (Singular-hyperbolic set)**.** *A compact invariant set of a vector field X is* singular-hyperbolic *if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction* [MPP1]*. A* singular-hyperbolic attractor *is an attractor which is also a singular-hyperbolic set.*

Singular-hyperbolic attractors cannot be hyperbolic and the most representative example is the geometric Lorenz [GW]. Our result is the following.

Theorem A. *Let U be an isolating block of a singular-hyperbolic attractor of X. If Y is a vector field C^r close to X, then $\{x \in U : \omega_Y(x)$ is singular$\}$ is residual in U*

This result is used to prove

Theorem B. *Singular-hyperbolic attractors with only one singularity in M are* persistent *as chain-transitive Lyapunov stable sets.*

The precise statement of Theorem B (including the definition of chain transitive set, Lyapunov stable set and persistence) will be given in Section 7. This paper is organized as follows. In Section 2 we give some preliminary lemmas. In particular, Lemma 2.1 introduces the *continuation* A_Y of an attracting set A for nearby vector fields Y. In Definition 2.3 we define *the region of weak attraction* $A_w(Z, C)$ of C, where C is a compact invariant sets of a vector field, as the set of points z such that $\omega_Z(z) \cap C \neq \emptyset$. Lemma 2.4 proves that if U is a neighborhood of C and $A_w(Z, C) \cap U$ is dense in U, then $A_w(Z, C) \cap U$ is residual in U. We finish this section with some elementary properties of the hyperbolic sets. We present two elementary properties of singular-hyperbolic attracting sets in Section 3.

In Section 4 we introduce *the Property (P)* for compact invariant sets C all of whose closed orbits are hyperbolic. It requires that the unstable manifold of every closed orbit in C intersect transversely the stable manifold of a singularity in C. This property has been proved for all singular-hyperbolic attractors Λ in [MPa1]. In Lemma 4.3 we prove that it is open, namely it holds for the continuation Λ_Y of Λ. The proof is similar to the one in [MPa1].

In Section 5 we study the topological dimension [HW] of the omega-limit sets in an isolating block U of a singular-hyperbolic attracting set with the Property (P). In particular, Theorem 5.2 proves that if $x \in U$ then the omega-limit set of x either contains a singularity or has topological dimension one provided the stable manifolds of the singularities in U do not intersect a neighborhood of x. The proof uses the methods in [M1] with the Property (P) playing the role of the transitivity. We need this theorem to apply the Bowen's theory of one-dimensional hyperbolic sets [Bo].

In Section 6 we prove Theorem A. The proof is based on Theorem 6.1 where it is proved that if U is an isolating block of a singular-hyperbolic attracting set with the Property (P) of a vector field Y, then $A_w(Y, Sing(Y, U)) \cap U$ is dense in U (here $Sing(Y, U)$ denotes the set of singularities of Y in U). The

proof follows applying the Bowen's theory (that can be used by Theorem 5.2) and the arguments in [MPa1, p. 371]. It will follow from Lemma 2.4 applied to $C = Sing(Y, U)$ that $A_w(Y, Sing(Y, U)) \cap U$ is residual in U. Theorem A follows because $\omega_Y(x)$ is singular $\forall x \in A_w(Y, Sing(Y, U)) \cap U$. In Section 7 we prove Theorem B (see Theorem 7.5).

2. Preliminary lemmas

We state some preliminary results. The first one claims a sort of stability of the attracting sets. It seems to be well known and we prove it here for completeness. If M is a manifold and $U \subset M$ we denote by $int(U)$ and $clos(U)$ the interior and the closure of U respectively.

Lemma 2.1 (Continuation of attracting sets)**.** *Let A be an attracting set* containing a hyperbolic closed orbit *of a C^r vector field X. If U is an isolating block of A, then for every vector field Y C^r close to X* the continuation

$$A_Y = \cap_{t\geq 0} Y_t(U)$$

of A in U *is an attracting set with isolating block U of Y.*

Proof. Since A contains a hyperbolic closed orbit we have that $A_Y \neq \emptyset$ for every Y close to X (use for instance the Hartman-Grobman Theorem [dMP]). Since U is compact we have that A_Y also does. Then, to prove the lemma, we only need to prove that if Y is close to X then U is a compact neighborhood of A_Y. For this we proceed as follows. Fix an open set D such that

$$A \subset D \subset clos(D) \subset int(U)$$

and for all $n \in I\!N$ we define

$$U_n = \cap_{t\in[0,n]} X_t(U).$$

Clearly U_n is a compact set sequence which is nested ($U_{n+1} \subset U_n$) and satisfies $A = \cap_{n\in I\!N} U_n$. Because U_n is nested we can find n_0 such that $U_{n_0} \subset D$. In other words

$$\cap_{t\in[0,n_0]} X_t(U) \subset D.$$

Taking complement one has

$$M \setminus D \subset \cup_{t\in[0,n_0]} X_t(M \setminus U).$$

But $X_t(M\setminus U)$ is open ($\forall t$) since U is compact and X_t is a diffeomorphism. Hence $\{X_t(M \setminus U) : t \in [0, n_0]\}$ is an open covering of $M \setminus D$. Because D is open we have that $M \setminus D$ is compact and so there are finitely many $t_1, \cdots, t_k \in [0, n_0]$ such that

$$M \setminus D \subset X_{t_1}(M \setminus U) \cup \cdots \cup X_{t_k}(M \setminus U).$$

By the continuous dependence of $Y_t(U)$ on Y (with t fixed) one has

$$M \setminus D \subset Y_{t_1}(M \setminus U) \cup \cdots \cup Y_{t_k}(M \setminus U)$$

for all Y C^r close to X. By taking complement once more we obtain

$$Y_{t_1}(U) \cap \cdots \cap Y_{t_k}(U) \subset D.$$

As $t_1, \cdots, t_k \geq 0$ one has $\cap_{t \in [0,n_0]} Y_t(U) \subset Y_{t_1}(U) \cap \cdots \cap Y_{t_k}(U)$ and then

$$\cap_{t \in [0,n_0]} Y_t(U) \subset D$$

for every Y close to X. On the other hand, it follows from the definition that $A_Y \subset \cap_{t \in [0,n_0]} Y_t(U)$ and so $A_Y \subset D$ for every Y close to X. Because $clos(D) \subset int(U)$ we have that $A_Y \subset int(U)$. This proves that U is a compact neighborhood of A_Y and the lemma follows. □

Remark 2.2. *The above proof shows that the compact set-valued map $Y \to A_Y$ is continuous in the following sense: For every open set D containing A one has $A_Y \subset D$ for every Y C^r close to X. Such a continuity is weaker than the continuity with respect to the Hausdorff metric. It follows from the above-mentioned continuity that if A is a singular-hyperbolic attracting set of X and Y is close to X, then the continuation A_Y in U is a singular-hyperbolic attracting set of Y.*

The following definition can be found in [BS, Chapter V].

Definition 2.3 (Region of attraction)**.** *Let C be a compact invariant set of a vector field Z. We define* the region of attraction *and* the region of weak attraction *of C by*

$$A(C) = \{x \in M : \omega_X(p) \subset C\} \quad \textit{and} \quad A_w(C) = \{z : \omega_Y(z) \cap C \neq \emptyset\}$$

respectively. We shall write $A(Z,C)$ and $A_w(Z,C)$ to indicate dependence on Z.

The region of attraction is also called *stable set*. The inclusion below is obvious

$$A(Z,C) \subset A_w(Z,C). \tag{1}$$

The elementary lemma below will be used in Section 6. Again we prove it for the sake of completeness.

Lemma 2.4. *If C a compact invariant set of a vector field Z and U is a compact neighborhood of C, then the following properties are equivalent:*
1. $A_w(Z,C) \cap U$ is dense in U
2. $A_w(Z,C) \cap U$ is residual in U.

Proof. Clearly (2) implies (1). Now we assume (1) namely $A_w(Z,C) \cap U$ is dense in U. Defining

$$W_n = \{x \in U : Z_t(x) \in B_{1/n}(C) \text{ for some } t > n\} \quad \forall n \in I\!N$$

one has

$$A_w(Z,C) \cap U = \cap_n W_n.$$

In particular $A_w(Z,C)\cap U\subset W_n$ for all n. Hence W_n is dense in U (for all n) since $A_w(Z,C)\cap U$ does. On the one hand, W_n is open in U [dMP, Tubular Flow-Box Theorem] because $B_{1/n}(T)$ is open. This proves that W_n is open-dense in U and the result follows. $\square$

Next we state the classical definition of hyperbolic set.

Definition 2.5 (Hyperbolic set). *A compact, invariant set H of a C^1 vector field X is* hyperbolic *if there are a continuous, tangent bundle, invariant, splitting $T\Lambda=E^s\oplus E^X\oplus E^u$ and positive constants C,λ such that $\forall x\in H$ one has:*
1. E^X_x is the direction of $X(x)$ in T_xM.
2. E^s is contracting: $\|DX_t(x)/E^s_x\|\leq Ce^{-\lambda t}$, $\forall t\geq 0$.
3. E^u is expanding: $\|DX_t(x)/E^u_x\|\geq C^{-1}e^{\lambda t}$, $\forall t\geq 0$.
A closed orbit of X is hyperbolic if it is hyperbolic as a compact, invariant set of X. A hyperbolic set is saddle-type if $E^s\neq 0$ and $E^u\neq 0$.

The Invariant Manifold Theory [HPS] says that through each point $x\in H$ pass smooth injectively immersed submanifolds $W^{ss}(x),W^{uu}(x)$ tangent to E^s_x,E^u_x at x. The manifold $W^{ss}(x)$, the strong stable manifold at x, is characterized by $y\in W^{ss}(x)$ if and only if $d(X_t(y),X_t(y))$ goes to 0 exponentially as $t\to\infty$. Similarly $W^{uu}(x)$, the strong unstable manifold at x, is characterized by $y\in W^{uu}(x)$ if and only if $d(X_t(y),X_t(x))$ goes to 0 exponentially as $t\to-\infty$. These manifolds are invariant, i.e. $X_t(W^{ss}(x))=W^{ss}(X_t(x))$ and $X_t(W^{uu}(x))=W^{uu}(X_t(x))$, $\forall t$. For all $x,x'\in H$ we have that $W^{ss}(x)$ and $W^{ss}(x')$ either coincides or are disjoint. The maps $x\in H\to W^{ss}(x)$ and $x\in H\to W^{uu}(x)$ are continuous (in compact parts). For all $x\in H$ we define

$$W^s_X(x)=\cup_{t\in I\!R}W^{ss}(X_t(x)) \text{ and } W^u_X(x)=\cup_{t\in I\!R}W^{uu}(X_t(x)).$$

Note that if $O\subset H$ is a closed orbit then

$$A(X,O)=W^s_X(O)$$

but $A_w(X,O)\neq W^s_X(O)$ in general. If H is saddle-type and $dim(M)=3$, then both $W^s_X(x),W^u_X(x)$ are one-dimensional submanifolds of M. In this case given $\epsilon>0$ we denote by $W^{ss}_X(x,\epsilon)$ an interval of length ϵ in $W^{ss}_X(x)$ centered at x (this interval is often called the local strong stable manifold of x).

Definition 2.6. *Let $\{O_n:n\in I\!N\}$ be a sequence of hyperbolic periodic orbits of X. We say that* the size of $W^s_X(O_n)$ is uniformly bounded away from zero *if there is $\epsilon>0$ such that the local strong stable manifold $W^{ss}_X(x_n,\epsilon)$ is well defined for every $x_n\in O_n$ and every $n\in I\!N$.*

Remark 2.7. *Let O_n be a sequence of hyperbolic periodic orbits of a vector field X. It follows from the Stable Manifold Theorem for hyperbolic sets* [HPS] *that the size of $W^s_X(O_n)$ is uniformly bounded away from zero if all the periodic orbits O_n ($n\in I\!N$) are contained in the same hyperbolic set H of X.*

3. Two lemmas for singular-hyperbolic attracting sets

Hereafter we denote by M a compact three manifold. Recall that $clos(\cdot)$ denotes the closure of $(\cdot)$. In addition, $B_\delta(x)$ denotes the (open) δ-ball in M centered at x. If $H \subset M$ we denote $B_\delta(H) = \cup_{x\in H} B_\delta(x)$. For every vector field X on M we denote by $Sing(X)$ the set of singularities of X and if $B \subset M$ we define $Sing(X, B) = Sing(X) \cap B$.

Lemma 3.1. *Let Λ be a singular-hyperbolic attracting set of a C^r vector field Z on M. Let U be an isolating block of Λ. If $x \in U$ and $\omega_Z(x)$ is non-singular, then every $k \in \omega_Z(x)$ is accumulated by a hyperbolic periodic orbit sequence $\{O_n : n \in \mathbb{N}\}$ such that the size of $W^s_Z(O_n)$ is uniformly bounded away from zero.*

Proof. For every $\epsilon > 0$ we define

$$\Lambda_\epsilon = \cap_{t\in\mathbb{R}} Z_t(\Lambda \setminus B_\epsilon(Sing(Z, \Lambda)).$$

Clearly Λ_ϵ is either $\emptyset$ or a compact, invariant, non-singular set of Z. If $\Lambda_\epsilon \neq \emptyset$, then Λ_ϵ is hyperbolic [MPP2]. Observe that $\omega_X(x)$ is non-singular by assumption. Then, there are $\epsilon > 0$ and $T > 0$ such that

$$Z_t(x) \notin clos(B_\epsilon(Sing(Z, U))), \quad \forall t \geq T.$$

It follows that $\omega_Z(x) \subset \Lambda_\epsilon$ and so $\Lambda_\epsilon \neq \emptyset$ is a hyperbolic set. In addition, for every $\delta > 0$ there is $T_\delta > 0$ such that

$$Z_t(x) \in B_\delta(\Lambda_\epsilon),$$

for every $t > T_\delta$. Pick $k \in \omega_Z(x)$. The last property implies that for every $\delta > 0$ there is a periodic δ-pseudo-orbit in $B_\delta(\Lambda_\epsilon))$ formed by paths in the positive Z-orbit of x. Applying the Shadowing Lemma for Flows [HK, Theorem 18.1.6 pp. 569] to the hyperbolic set Λ_ϵ we arrange a periodic orbit sequence $O_n \subset \Lambda_{\epsilon/2}$ accumulating k. Then, Remark 2.7 applies since $H = \Lambda_{\epsilon/2}$ is hyperbolic and contains O_n (for all n). The lemma is proved. □

The following is a minor modification of [M2, Theorem A].

Lemma 3.2. *If U is an isolating block of a singular-hyperbolic attractor of a C^r vector field X in M, then every attractor in U of every vector field C^r close to X is singular.*

Proof. Let Λ be the singular-hyperbolic attractor of X having U as isolating block. By [M2, Theorem A] there is a neighborhood D of Λ such that every attractor of every vector field Y C^r close to X is singular. By Remark 2.2 we have that $\cap_{t\geq 0} Y_t(U) \subset D$ for all Y close to X. Now if $A \subset U$ is an attractor of Y, then $A \subset \cap_{t\geq 0} Y_t(U)$ by invariance. We conclude that $A \subset D$ and then A is singular for all Y close to X. This proves the lemma. □

4. Property (P)

First we state the definition. As usual we write $S \pitchfork S' \neq \emptyset$ to indicate that there is a transverse intersection point between the submanifolds S, S'.

Definition 4.1 (The Property (P))**.** *Let Λ be a compact invariant set of a vector field X. Suppose that all the closed orbits of Λ are hyperbolic. We say that Λ satisfies* the Property (P) *if for every point p on a periodic orbit of Λ there is $\sigma \in Sing(X, \Lambda)$ such that*

$$W^u_Y(p) \pitchfork W^s_Y(\sigma) \neq \emptyset.$$

The lemma below is a direct consequence of the classical Inclination-lemma [dMP] and the transverse intersection in Property (P).

Lemma 4.2. *Let Λ a compact invariant set with the Property (P) of a vector field Z in a manifold M and I be a submanifold of M. If there is a periodic orbit $O \subset \Lambda$ of Z such that*

$$I \pitchfork W^s_Z(O) \neq \emptyset,$$

then

$$I \cap \left(\cup_{\sigma \in Sing(Z,\Lambda)} W^s_Z(\sigma)\right) \neq \emptyset.$$

The Property (P) was proved in [MPa1, Theorem 5.1] for all singular-hyperbolic attractors. Here we prove that such a property is open, namely it holds for the continuation in Lemma 2.1 of a singular-hyperbolic attractor.

Lemma 4.3 (Openness of the Property (P))**.** *Let U be an isolating block of a singular-hyperbolic attractor of a C^r vector field X on M. Then, the continuation*

$$\Lambda_Y = \cap_{t \geq 0} Y_t(U)$$

has the Property (P) for every vector field Y C^r close to X.

Proof. By Lemma 2.1 we have that Λ_Y is an attracting set with isolating block U since Λ has a hyperbolic singularity. Now let p be a point of a periodic orbit $\gamma \subset \Lambda_Y$ of Y. Then

$$clos(W^u_Y(p)) \subset \Lambda_Y$$

since Λ_Y is attracting. We claim

$$clos(W^u_Y(p)) \cap Sing(Y, U) \neq \emptyset.$$

Indeed suppose that it is not so, i.e. there is Y C^r close to X such that $clos(W^u_Y(p)) \cap Sing(Y, U) = \emptyset$ for some p in a periodic orbit of Y in U. It follows from [MPP2] that $clos(W^u_Y(p))$ is a hyperbolic set. Since $W^u_Y(p)$ is a two-dimensional submanifold we can easily prove that $clos(W^u_Y(p))$ is an attracting set of Y. This attracting set necessarily contains a hyperbolic attractor A of Y. Since $A \subset clos(W^u_Y(p)) \subset \Lambda_Y \subset U$ we conclude that $A \subset U$. By Lemma 3.2 we have that A is singular as well. We conclude that A is an attracting singularity of Y in U. This contradicts the volume expanding condition at Definition 1.4 and

the claim follows. One completes the proof of the lemma using the claim as in [MPa1, Theorem 5.1]. □

5. Topological dimension and the Property (P)

We study the topological dimension of the omega-limit set in an isolating block of a singular-hyperbolic attracting set with the Property (P). First of all we recall the classical definition of topological dimension [HW].

Definition 5.1. *The* topological dimension *of a space E is either -1 (if $E=\emptyset$) or the last integer k for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than k. A space with topological dimension k is said to be* k-dimensional*.*

The result of this section is the following.

Theorem 5.2. *Let U be an isolating block of a singular-hyperbolic attracting set with the Property (P) of a C^r vector field Y on M. If $x \in U$ and there is $\delta > 0$ such that*

$$B_\delta(x) \cap \left(\cup_{\sigma \in Sing(Y,U)} W^s_Y(\sigma)\right) = \emptyset,$$

then $\omega_Y(x)$ is either singular or a one-dimensional hyperbolic set.

Proof. Let Λ_Y be the singular-hyperbolic attracting set of Y having U as isolating block. Obviously $Sing(Y,U) = Sing(Y,\Lambda_Y)$. Let x, δ be as in the statement. Define

$$H = \omega_Y(x).$$

We shall assume that H is non-singular. Then H is a hyperbolic set by [MPP2]. To prove that H is one-dimensional we shall use the arguments in [M1]. However we have to take some care because Λ is not transitive. The Property (P) will supply an alternative argument. Let us present the details.

First we note that by Lemma 3.1 every point $k \in H$ is accumulated by a periodic orbit sequence O_n satisfying the conclusion of that lemma. Second, by the Invariant Manifold Theory [HPS], there is an invariant contracting foliation $\{\mathcal{F}^s(w) : w \in \Lambda_Y\}$ which is tangent to the contracting direction of Y in Λ_Y. A cross-section of Y will be a 2-disk transverse to Y. When $w \in \Lambda_Y$ belongs to a 2-disk D transverse to Y, we define $\mathcal{F}^s(w, D)$ as the connected component containing w of the projection of $\mathcal{F}^s(w)$ onto D along the flow of Y. The boundary and the interior of D (as a submanifold of M) are denoted by ∂D and $int(D)$ respectively. D is a *rectangle* if it is diffeomorphic to the square $[0,1]\times[0,1]$. In this case ∂D as a submanifold of M is formed by four curves $D^t_h, D^b_h, D^l_v, D^r_v$ (v for vertical, h for horizontal, l for left, r for right, t for top and b for bottom). One defines vertical and horizontal curves in D in the natural way.

Now we prove a sequence of lemmas corresponding to lemmas 1-4 in [M1] respectively.

Lemma 5.3. *For every regular point $z \in \Lambda_Y$ of Y there is a rectangle Σ such that the properties below hold:*
1. $z \in int(\Sigma)$;
2. If $w \in \Lambda_Y$ then $\mathcal{F}^s(w, \Sigma)$ is a horizontal curve in Σ;
3. If $\Lambda_Y \cap \Sigma_h^t \neq \emptyset$ then $\Sigma_h^t = \mathcal{F}^s(w, \Sigma)$ for some $w \in \Lambda_Y \cap \Sigma$;
4. If $\Lambda_Y \cap \Sigma_h^b$ then $\Sigma_h^b = \mathcal{F}^s(w, \Sigma)$ for some $w \in \Lambda_Y \cap \Sigma$.

Proof. The proof of this lemma is similar to [M1, Lemma 1]. Observe that the corresponding proof in [M1] does not use the transitivity hypothesis. □

Definition 5.4. *If $w \in H \cap \Sigma$ we denote by $(H \cap \Sigma)_w$ the connected component of $H \cap \Sigma$ containing w.*

With this definition we shall prove the following lemma.

Lemma 5.5. *If $w \in H \cap \Sigma$ and $(H \cap \Sigma)_w \neq \{w\}$, then $(H \cap \Sigma)_w$ contains a non-trivial curve in the union $\mathcal{F}^s(w, \Sigma) \cup \partial\Sigma$.*

Proof. We follow the same steps of the proof of Lemma 2 in [M1]. First we observe that $(H \cap \Sigma)_x \cap (int(\Sigma) \setminus \mathcal{F}^s(x, \Sigma)) \neq \emptyset$. Hence we can fix $w' \in (H \cap \Sigma)_x \cap (int(\Sigma) \setminus \mathcal{F}^s(x, \Sigma))$. Clearly $\mathcal{F}^s(w', \Sigma)$ is a horizontal curve which together with $\mathcal{F}^s(w, \Sigma)$ form the horizontal boundary curves of a rectangle R in Σ. One has that $H \cap int(B) \neq \emptyset$ for, otherwise, w and w' would be in different connected components of $H \cap \Sigma$ a contradiction. Hence we can choose $h \in H \cap int(B)$. Since $H = \omega_Y(y)$ we have that there is y' in the positive Y-orbit of y arbitrarily close to h. In particular, $y' \in int(B)$. By the continuity of the foliation $\mathcal{F}^s$ we have that $\mathcal{F}^s(y', \Sigma)$ is a horizontal curve separating Σ in two connected components containing w and w' respectively. Since w, w' belong to the same connected component of $H \cap \Sigma$ we conclude that there is $k \in \mathcal{F}^s(y', \Sigma) \cap H \neq \emptyset$.

On one hand, by Lemma 3.1, $k \in H$ is accumulated by a hyperbolic periodic orbit sequence O_n such that the size of $W^s_Y(O_n)$ is uniform bounded away from zero. On the other hand y' belongs to the positive orbit of y and $y \in B_\delta(x)$. By the uniform size of $W^s_Y(O_n)$ one has $B_\delta(x) \cap W^s_Y(O_n) \neq \emptyset$ for some $n \in I\!N$. Since $B_\delta(x)$ is open we conclude that

$$B_\delta(x) \pitchfork W^s_Y(O_n) \neq \emptyset$$

Then,

$$B_\delta(x) \cap \left(\cup_{\sigma \in Sing(Y,U)} W^s_Y(\sigma)\right) \neq \emptyset$$

by Lemma 4.2 since Λ_Y has the Property (P). This is a contradiction which proves the lemma. □

Lemma 5.6. *For every $w \in H$ there is a rectangle Σ_w containing w in its interior such that $H \cap \Sigma_w$ is 0-dimensional.*

Proof. This lemma corresponds to Lemma 3 in [M1] with similar proof. Let $\Sigma_w = \Sigma$ where Σ is given by Lemma 5.5. Let $J \subset \mathcal{F}^s(w, \Sigma) \cap \partial\Sigma$ be the curve

in the conclusion of this lemma. We can assume that J is contained in either $\mathcal{F}^s(w,\Sigma)$ or $\partial\Sigma$. If $J \subset \mathcal{F}^s(w,\Sigma)$ we can prove as in the proof of [M3, Lemma 3] that $y \in H$ and so y is accumulated by periodic orbits whose unstable and stable manifolds have uniform size. We arrive a contradiction by Lemma 4.3 as in the last part of the proof of Lemma 5.5. Hence we can assume that $J \subset \partial\Sigma$. We can further assume that $J \subset \Sigma_v^l$ (say) for otherwise we get a contradiction as in the previous case. Now if $J \subset \Sigma_v^l$ then we can obtain a contradiction as before again using the Property (P) and Lemma 4.2. This proves the result. □

The following lemma corresponds to [M1, Lemma 4].

Lemma 5.7. *H can be covered by a finite collection of closed one-dimensional subsets.*

Proof. If $w \in H$ we consider the cross-section Σ_w in Lemma 5.7. By saturating forward and backward Σ_w by the flow of Y we obtain a compact neighborhood of w which is one-dimensional (see [HW, Theorem III 4 p. 33]). Hence there is a neighborhood covering of H by compact one-dimensional sets. Such a covering has a finite subcovering since H is compact. Such a subcovering proves the result. □

Theorem 5.2 now follows from Lemma 5.7 and [HW, Theorem III 2 p. 30]. □

6. Proof of Theorem A

The proof is based on the following result.

Theorem 6.1. *Let U be an isolating block of a singular-hyperbolic attracting set with the Property (P) of a vector field Y on M. Then $A_w(Y, Sing(Y,U)) \cap U$ is residual in U.*

Proof. By Lemma 2.4 it suffices to prove that $A_w(Y, Sing(Y,U)) \cap U$ is dense in U. Let Λ_Y be the singular-hyperbolic attracting set of Y having U as isolating block. Obviously $Sing(Y,U) = Sing(Y,\Lambda_Y)$. To simplify the notation we write $R_Y = A_w(Y, Sing(Y,U)) \cap U$. Suppose by contradiction that R_Y is not dense in U. Then, there is $x \in U$ and $\delta > 0$ such that $B_\delta(x) \cap R_Y = \emptyset$. In particular, $\omega_Y(x) \cap Sing(Y,U) = \emptyset$ and so $\omega_Y(x)$ is non-singular. Recalling the inclusion Eq.(1) at Section 2 one has

$$U \cap \left(\cup_{\sigma \in Sing(Y,U)} W^s_Y(\sigma)\right) \subset R_Y.$$

Thus

$$B_\delta(x) \cap \left(\cup_{\sigma \in Sing(Y,U)} W^s_Y(\sigma)\right) = \emptyset. \tag{2}$$

It then follows from Theorem 5.2 that $H = \omega_Y(x)$ is a one-dimensional hyperbolic set. This allows to apply the Bowen's Theory [Bo] of one-dimensional hyperbolic sets. More precisely there is a family of (disjoint) cross-sections $\mathcal{S} = \{S_1, \cdots, S_r\}$ of small diameter such that H is the flow-saturated of $H \cap int(\mathcal{S}')$, where $\mathcal{S}' = \cup S_i$

and $int(\mathcal{S}')$ denotes the interior of $\mathcal{S}'$ (as a submanifold). Next we choose an interval I tangent to the central direction E^c of Y in U such that

$$x \in I \subset B_\delta(x).$$

We choose I to be transverse to the direction E^Y induced by Y. Since E^c is volume expanding and H is non-singular we have that the Poincaré map induced by X on $\mathcal{S}'$ is expanding along I. As in [MPa1, p. 371] we can find $\delta' > 0$ and a open arc sequence $J_n \subset \mathcal{S}'$ in the positive orbit of I with length $\geq \delta'$ such that there is x_n *in the positive orbit of* x contained in the interior of J_n. We can fix $S = S_i \in \mathcal{S}$ in order to assume that $J_n \subset S$ for every n. Let $w \in S$ be a limit point of x_n. Then $w \in H \cap int(\mathcal{S}')$. Because I is tangent to E^c the interval sequence J_n converges to an interval $J \subset W^u_Y(w)$ in the C^1 topology ($W^u_Y(w)$ exists because $w \in H$ and H is hyperbolic). J is not trivial since the length of J_n is $\geq \delta'$. It follows from this lower bound that J_n intersects $W^s_Y(w)$ for some n large. Now, by Lemma 3.1, w is accumulated by periodic orbits O_n satisfying the conclusion of this lemma. The continuous dependence in compact parts of the stable manifolds implies $J_n \pitchfork W^s_Y(O_n) \neq \emptyset$. Since J_n is in the positive orbit of I and $I \subset B_\delta(x)$ we obtain

$$B_\delta(x) \pitchfork W^s_Y(O_n) \neq \emptyset.$$

Then,

$$B_\delta(x) \cap \left(\cap_{\sigma \in Sing(Y,U)} W^s_Y(\sigma)\right) \neq \emptyset$$

by Lemma 4.2 since Λ_Y has the Property (P). This is a contradiction by Eq.(2). This contradiction proves that R_Y is dense in U for all Y C^r close to X. $\square$

Proof of Theorem A: Let U be an isolating block of a singular-hyperbolic attractor of a C^r vector field X on M. By Lemma 2.1 we have that $\Lambda_Y = \cap_{t\geq 0} Y_t(U)$ is a singular-hyperbolic attracting set with isolating block U for all vector field Y C^r close to X. In addition, Λ_Y has the Property (P) by Lemma 4.3. It follows from Theorem 6.1 that $A_w(Y, Sing(Y,U)) \cap U$ is residual in U. The result follows because $\omega_Y(x)$ is singular $\forall x \in A_w(Y, Sing(Y,U)) \cap U$ (recall Definition 2.3). $\square$

Remark 6.2. *Let Y be a vector field in a manifold M. In* [BS, Chapter V] *it was defined a* weak attractor *of Y as a closed set $C \subset M$ such that $A_w(Y,C)$ is a neighborhood of C. Similarly one can define a* generic weak attractor *of Y as a closed set $C \subset M$ such that $A(Y,C) \cap U$ is residual in U for some neighborhood U of C (compare with the definition of generic attractor* [Mi, Appendix 1 p.186]*). A direct consequence of Theorem 6.1 is that the set of singularities of a singular-hyperbolic attractor of Y is a generic weak attractor of Y.*

7. Persistence of singular-hyperbolic attractors

In this section we prove Theorem B as an application of Theorem A. The idea is to address the question below which is a weaker local version of the Palis's conjecture [P].

Question 7.1. *Let Λ an attractor of a C^r vector field X on M and U be an isolating block of Λ. Does every vector field C^r close to X exhibit an attractor in U?*

This question has positive answer for hyperbolic attractors, the geometric Lorenz attractors and the example in [MPu]. In general we give a partial positive answer for all singular-hyperbolic attractors with only one singularity in terms of chain-transitive Lyapunov stable sets.

Definition 7.2. *A compact invariant set Λ of a vector field X is* Lyapunov stable *if for every open set $U \supset \Lambda$ there is an open set $\Lambda \subset V \subset U$ such that $\cup_{t>0} X_t(V) \subset U$.*

Recall that $B_\delta(x)$ denotes the (open) ball centered at x with radius $\delta > 0$.

Definition 7.3. *Given $\delta > 0$ we define a δ-*chain *of X as a pair of finite sequences $q_1, ..., q_{n+1} \in M$ and $t_1, ..., t_n \geq 1$ such that*

$$X_{t_i}(B_\delta(q_i)) \cap B_\delta(q_{i+1}) \neq \emptyset, \quad \forall i = 1, \cdots, n.$$

The δ-chain joints p, q if $q_1 = q$ and $q_{n+1} = p$. A compact invariant set Λ of X is chain-transitive *if every pair of points $p, q \in \Lambda$ can be joined by a δ-chain, $\forall \delta > 0$.*

Every attractor is a chain-transitive Lyapunov stable set but not vice versa. The following generalizes the concept of robust transitive attractor (see for instance [MPa4]).

Definition 7.4. *Let Λ be a chain-transitive Lyapunov stable set of a C^r vector field X, $r \geq 1$. We say that Λ is C^r* persistent *if for every neighborhood U of Λ and every vector field Y C^r close to X there is a chain-transitive Lyapunov stable set Λ_Y of Y in U such that $A(Y, \Lambda_Y) \cap U$ is residual in U.*

Compare this definition with the one in [Hu] where it is required the continuity of $Y \to \Lambda_Y$ (with respect to the Hausdorff metric) instead of the residual condition of the stable set. Another related definition is that of C^r weakly robust attracting sets in [CMP]. The result of this section is the following one. It is precisely the Theorem B stated in the Introduction.

Theorem 7.5. *Singular-hyperbolic attractors with only one singularity for C^r vector fields on M are C^r persistent.*

Proof. Let Λ be a singular-hyperbolic attractor of a C^r vector field X on M. Suppose that Λ contains a unique singularity σ. Let U be a neighborhood of Λ.

We can suppose that U is an isolating block. Let $\sigma(Y)$ the continuation of σ for every vector field Y close to X. Note that $\sigma(X) = \sigma$. Clearly $Sing(Y, U) = \{\sigma(Y)\}$ for every Y close to X.

For every vector field Y C^r close to X one defines

$$\Lambda(Y) = \{q \in \Lambda_Y : \forall \delta > 0 \ \exists \delta\text{-chain joining } \sigma(Y) \text{ and } q\}.$$

Recall that Λ_Y is the continuation of Λ in U for Y close to X as in Lemma 2.1. We note that $\Lambda(Y) \neq \Lambda_Y$ in general [MPu].

To prove the theorem we shall prove that $\Lambda(Y)$ satisfies the following properties ($\forall Y$ C^r close to X):
(1) $\Lambda(Y)$ is Lyapunov stable.
(2) $\Lambda(Y)$ is chain-transitive.
(3) $A(Y, \Lambda(Y)) \cap U$ is residual in U.

One can easily prove (1). To prove (2) we pick $p, q \in \Lambda(Y)$ for Y close to X and fix $\delta > 0$. By Theorem A there is $x \in B_\delta(p)$ such that $\omega_Y(x)$ contains $\sigma(Y)$. Hence there is $t > 1$ such that $X_t(x) \in B_\delta(\sigma)$. On the other hand, since $q \in \Lambda(Y)$, there is a δ-chain $(\{t_1, \cdots, t_n\}, \{q_1, \cdots, q_{n+1}\})$ joining σ to q. Then (2) follows since the δ-chain $(\{t, t_1, \cdots, t_n\}, \{x, q_1, \cdots, q_{n+1}\})$ joints p and q. To finish we prove (3). It follows from well known properties of Lyapunov stable sets [BS] that $\Lambda(Y) = \cap_n O_n$ where O_n is a nested sequence of positively invariant open sets of Y. Obviously we can assume that $O_n \subset U$ for all n. Clearly the stable set of O_n is open in U. Let us prove that such a stable set is dense in U. Let O be an open subset of U. By Theorem 5.2 there is $x \in O$ such that $\omega_Y(x)$ contains $\sigma(Y)$. Clearly $\sigma(Y)$ belongs to O_n and so $\omega_Y(x)$ intersects O_n as well. Hence there is $t > 0$ such that $X_t(x) \in O_n$. The last implies that x belongs to the stable set of O_n. This proves that the stable set of O_n is dense for all n. But the stable set of $\Lambda(Y)$ is the intersection of $W^s_Y(O_n)$ which is open-dense in U. We conclude that the stable set of $\Lambda(Y)$ is residual and the proof follows. $\square$

Theorem 7.5 gives only a partial answer for Question 7.1 (in the one singularity case) since chain-transitive Lyapunov stable set are not attractors in general. However a positive answer for the question will follow (in the one singularity case) once we give positive answer for the questions below.

Question 7.6. *Is a singular-hyperbolic, Lyapunov stable set an attracting set?*

Question 7.7. *Is a singular-hyperbolic, chain-transitive, attracting set a transitive set?*

As it is well known these questions have positive answer replacing singular-hyperbolic by hyperbolic in their corresponding statements. Besides it, a positive answer for Question 7.6 holds provided the two branches of the unstable manifold of every singularity of the set are dense on the set [MPa3].

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